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Making Change Like Frobenius

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September 24, 2006

Let's imagine that we introduce a new coin system. Instead of using pennies, nickels, dimes, and quarters, let's say we agree on using only 4-cent and 7-cent coins. One might point out the following flaw of this new system: certain amounts cannot be exchanged, for example, 1, 2, or 5 cents. On the other hand, this deficiency makes our new coin system more interesting than the old one, because we can ask the question: "which amounts can be changed?" We will see shortly that there are only finitely many integer amounts that *cannot* be exchanged using our new coin system. A natural question, first tackled by Ferdinand Georg Frobenius and James Joseph Sylvester in the nineteenth century, is: "what is the *largest* amount that cannot be exchanged?" As mathematicians, we like to keep questions as general as possible, and so we ask: given coins of denominations a and b—positive integers without a common factor—can you give a formula $g(a,b)$ for the largest amount that cannot be exchanged using the coins a and b? This problem and its generalization for coins $a_1, a_2, \ldots, a_n$ is known as the *Frobenius coin-exchange problem.* To study the Frobenius number $g(a,b)$, we use the *Euclidean Algorithm.* For integers a and b that have no common factor, this algorithm yields integers x and y such that $ax + by = 1$.

Problems

1. Find $g(4,7)$ and $g(5,11)$.

2. Find x and y such that $4x + 7y = 1$. Find another x and y such that $4x + 7y = 1$.

3. Find x and y such that $5x + 11y = 1$. Find x and y such that $5x + 11y = 39$.

4. Show that, if t is a given integer, we can always find integers x and y such that $4x + 7y = t$. Generalize to two coins a and b with no common factor.

5. Show that, if t is a given integer, we can always find integers x and y such that $4x + 7y = t$ and $0 \le x \le 6$. Generalize to two coins a and b with no common factor.

6. Show that the following recipe for determining whether or not a given amount t can be changed (using the coins 4 and 7) works: Given t, find integers x and y such that $4x + 7y = t$ and $0 \le x \le 6$. Then t can be changed precisely if $y \ge 0$. Generalize to two coins a and b with no common factor.

7. Use the previous argument to re-compute $g(4,7)$. Generalize your argument to compute $g(a,b)$, for any two coins a and b with no common factor.

8. Suppose t is an integer between 1 and $ab - 1$ that is not a multiple of a or b. Prove that if the amount t can be changed then $ab - t$ cannot be changed, and converseley, if t cannot be changed then $ab - t$ can be changed.

9. Prove that there are $\frac{1}{2}(a-1)(b-1)$ amounts that cannot be changed.

10. Think about why $g(a,b)$ actually exists, if a and b have no common factor. More generally, prove that the general Frobenius problem is well defined. That is, show that, given $a_1, a_2, \ldots, a_d$ with no common factor, every sufficiently large integer is representable (in terms of $a_1, a_2, \ldots, a_d$).

11. Next week we will study the counting sequence

$$r_k = \# \left\{ (m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ ma + nb = k \right\} .$$

In words, r_k counts the representations of $k \in \mathbb{Z}_{\geq 0}$ as nonnegative linear combinations of a and b. The Frobenius problem asks for the largest among the r_k's that is 0. Prove that $r_{k+ab} = r_k + 1$.

A few remarks

The simple-looking formula for $g(a,b)$ that you have found in () inspired a great deal of research into formulas for the general Frobenius number $g\left(a_1, a_2, \ldots, a_d\right)$, with limited success: While it is safe to assume that the formula for $g(a,b)$ has been known for more than a century, no analogous formula exists for $d \geq 3$. The case $d = 3$ is solved algorithmically, i.e., there are efficient algorithms to compute $g(a,b,c)$ [2, 4, 5], and in form of a semi-explicit formula [3, 7]. The Frobenius problem for fixed $d \geq 4$ has been proved to be computationally feasible [1, 6], but not even an efficient practical algorithm for $d = 4$ is known. The formula in () is due to Sylvester and was published as a math problem in the *Educational Times* more than a century ago [9]. For more on the Frobenius problem, we refer to the research monograph [8]; it includes more than 400 references to articles written about the Frobenius problem.

References

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