



1. Prove that the sum of the degrees of the vertices of any finite graph is even.
2. Show that every simple graph has two vertices of the same degree.
3. Show that if n people attend a party and some shake hands with others (but not with themselves), then at the end, there are at least two people who have shaken hands with the same number of people.
4. Prove that a complete graph with n vertices contains $n(n - 1)/2$ edges.
5. Prove that a finite graph is bipartite if and only if it contains no cycles of odd length.
6. Show that if every component of a graph is bipartite, then the graph is bipartite.
7. Prove that if u is a vertex of odd degree in a graph, then there exists a path from u to another vertex v of the graph where v also has odd degree.
8. If the distance $d(u, v)$ between two vertices u and v that can be connected by a path in a graph is defined to be the length of the shortest path connecting them, then prove that the distance function satisfies the triangle inequality: $d(u, v) + d(v, w) \geq d(u, w)$.
9. Show that any graph where the degree of every vertex is even has an Eulerian cycle. Show that if there are exactly two vertices a and b of odd degree, there is an Eulerian path from a to b . Show that if there are more than two vertices of odd degree, it is impossible to construct an Eulerian path.
10. Show that in a directed graph where every vertex has the same number of incoming as outgoing paths there exists an Eulerian path for the graph.
11. Consider the sequence 01110100 as being arranged in a circular pattern. Notice that every one of the eight possible binary triples: 000, 001, 011, \dots , 111 appear exactly once in the circular list. Can you construct a similar list of length 16 where all the four binary digit patterns appear exactly once each? Of length 32 where all five binary digit patterns appear exactly once?
12. An n -cube is a cube in n dimensions. A cube in one dimension is a line segment; in two dimensions, it's a square, in three, a normal cube, and in general, to go to the next dimension, a copy of the cube is made and all corresponding vertices are connected. If we consider the cube to be composed of the vertices and edges only, show that every n -cube has a Hamiltonian circuit.
13. Show that a tree with n vertices has exactly $n - 1$ edges.

14. If u and v are two vertices of a tree, show that there is a unique path connecting them.
15. Show that any tree with at least two vertices is bipartite.
16. (The Schröder-Bernstein Theorem) Show that if set A can be mapped 1 – 1 onto a subset of B and B can be mapped 1 – 1 onto a subset of A , then sets A and B have the same cardinality. (Two sets have the same cardinality if there exists a 1 – 1 and onto mapping between them.)
17. Solve Instant Insanity.

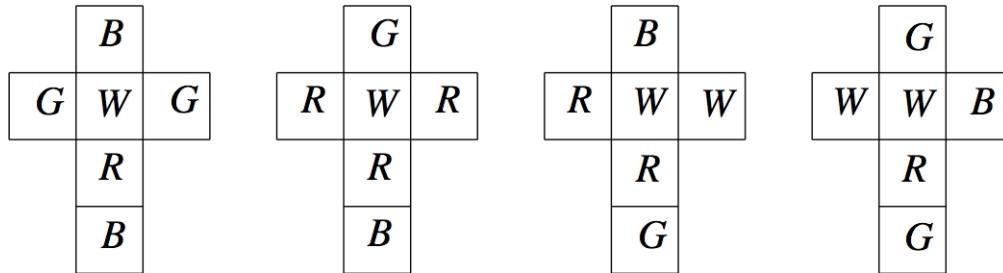


Figure 1: Instant Insanity Blocks

Figure 1 shows four unwrapped cubes that form the instant insanity puzzle. The letters “R”, “W”, “B” and “G” stand for the colors “red”, “white”, “blue” and “green”. The object of the puzzle is to stack the blocks in a pile of 4 in such a way that each of the colors appears exactly once on each of the four sides of the stack.